1 Notes on kinematics

1.1 Notation in cartesian and light-cone coordinates

Let us first clarify notation. The gluon 4-momentum takes the form

$$k^\mu = (\omega, k_l, k_T) \ , \tag{1}$$

where ω is the gluon energy, k_l its longitudinal momentum and k_T its transverse momentum. The index μ runs over 0, two transverse components T and L. In light-cone coordinates, we write

$$k^\nu = \left[k^+, k^-, k_T\right] \ , \tag{2}$$

where the index runs over $\nu = +, -, T$ and we have the definitions

$$k^+ \equiv \omega + k_l \, , k^- \equiv \omega - k_l \, . \tag{3}$$

Note that this implies $k_l \geq 0$ (for $k_l < 0$, the definition of k^+ and k^- is interchanged).

Constraint from masslessness: The gluon is massless in our calculations, and this implies $k^\nu k_\nu = k^+ k^- - k_T^2 = 0 = \omega^2 - k_l^2 - k_T^2$. This means that out of the three kinematic variables ω, k_l, k_T (or k^+, k^-, k_T), only two are independent. In light-cone coordinates, the dependent one is chosen to be

$$k^- = \frac{k_\perp^2}{k_+} \, . \tag{4}$$

leading to

$$k^\nu = \left[k^+, \frac{k_\perp^2}{k_+}, k_T\right] \ , \tag{5}$$

In cartesian coordinates, the mass-shell constraint can be written as

$$\omega = \sqrt{k_T^2 + k_l^2} \, . \tag{6}$$

Defining momentum fractions: We now introduce the energy fraction x, which the gluon carries away from a parent parton of energy E

$$x \equiv \frac{\omega}{E} \, . \tag{7}$$

We also define the fraction of light-cone +-momentum, which the gluon carries away from a parent parton of corresponding +-momentum E^+,

$$\tilde{x} \equiv \frac{k^+}{E^+} \, . \tag{8}$$

2 Kinematic constraints on k_T from mass-shell condition

For cartesian coordinates, the constraint on k_T follows directly from equation (6). For fixed gluon energy, the transverse momentum is maximal if k_l is minimal, and hence

$$k_T^{\mathrm{max,cartesian}} = \left.\sqrt{\omega^2 - k_l^2}\right|_{k_l=0} = \omega = x\,E\,. \qquad (9)$$

In light-cone coordinates, we have used up the mass-shell condition already in writing (5). So, from this expression alone, no constraint on a limit of k_T can be derived. To derive a limit, let us reintroduce knowledge about longitudinal momentum. We know $k^- = \omega - k_l = (\omega + k_l) - 2k_l = \tilde{x}E^+ - 2k_l$ and hence

$$0 = k^+k^- - k_T^2 = (\tilde{x}E^+)^2 - 2\tilde{x}E^+\,k_l - k_T^2 \qquad (10)$$

from which we find

$$k_T = \tilde{x}E^+\sqrt{1 - \frac{2\,k_l}{\tilde{x}E^+}} \qquad (11)$$

Now, we apply the same logic as for the Cartesian upper bound (9), to write

$$k_T^{\mathrm{max,lightcone}} = \left.\tilde{x}E^+\sqrt{1 - \frac{2\,k_l}{\tilde{x}E^+}}\right|_{k_l=0} = \left.\tilde{x}E^+\right|_{k_l=0}\,. \qquad (12)$$

Conclusions Note that we can rewrite equation (12) as

$$\left.\tilde{x}E^+\right|_{k_l=0} = \left.k^+\right|_{k_l=0} = \omega = x\,E\,. \qquad (13)$$

To arrive at the last expression, we have used the definition of k^+. This shows that **the constraints (9) in Cartesian coordinates and the constraint (12) in light-cone coordinates is exactly the same**, as it should be.

Now let us use $E^+ = 2\,E$. This implies

$$x = 2\,\tilde{x}\,. \qquad (14)$$

This is consistent with the notes of William and of Ulrich. In these notes, a second phase space constraint is explored, which gives an extra reduction factor $(1 - \tilde{x})$ at large $\tilde{x}$. For small $\tilde{x}$, however, equation (14) is consistent with the notes of William and Ulrich. In particular, their plot of $x(\tilde{x})$ has the correct slope factor 2 for small $\tilde{x}$.

3 Does the integrand depend on x or $\tilde{x}$?

If consistently used, both Cartesian coordinates and light-cone coordinates must yield the same result. For this, it is important that WHDG uses radiation cross sections which are written as functions of light-cone $\tilde{x}$, while ASW opacity approximation uses radiation cross sections written as function of Cartesian x.[1] However, the functional shapes of the integrands of WHDG and ASW are the same, and this cannot be true if x does not equal $\tilde{x}$.

So, which x is actually in the integrands? To find the simplest example, we can look e.g. for an expression to zeroth order in opacity, that means vacuum radiation. In this case, we can check with well-established perturbative expressions. Djordjevic + Gyulassy (see eq. 9 of nucl-th/0310076) write the gluon radiation spectrum in the vacuum (and for vanishing gluon mass) as

$$\omega \frac{dN_g^0}{d^3k} \propto \frac{k_T^2}{\left(k_T^2 + x^2\, M^2\right)^2} \,. \tag{15}$$

This expression is also used subsequently by WDHG. The question is whether this equation is written in terms of x, or whether x should be replaced by the light-cone fraction $\tilde{x}$. To check, I compare with the LO perturbative result for the dead-cone radiation spectrum, which is (see e.g. equation 13 of Kharzeev + Dokshitzer, hep-ph/0106202)

$$\propto \frac{k_T^2}{\left(k_T^2 + \omega^2\, \frac{M^2}{E^2}\right)^2} = \frac{k_T^2}{\left(k_T^2 + x^2\, M^2\right)^2} \,, \tag{16}$$

where ω and E are Cartesian energies of the gluon and the parent quark, and x is the Cartesian energy fraction.

Does comparison of (16) and (15) indicate that the integrands used by WDHG are written in Cartesian x-coordinates? Or is the difference between Cartesian and light-cone coordinates not within the accuracy of the calculation? These are the two logical possibilities. I note that ASW obtain equation (15) in terms of Cartesian coordinates, which is clearly consistent with the vacuum result.

[1] Let me emphasize again that the opacity approximation of ASW was never used for phenomenological comparisons, but this is another story.